

MODIFICATIONS OF TUTTE-GROTHENDIECK INVARIANTS AND TUTTE POLYNOMIALS

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ABSTRACT. We transform Tutte-Grothendieck invariants thus also Tutte polynomials on matroids so that the contraction-deletion rule for loops (isthmuses) coincides with the general case.

1. INTRODUCTION

A *Tutte-Grothendieck invariant* (shortly a *T-G invariant*) Φ is a mapping from the class of finite matroids to a commutative ring $(R, +, \cdot, 0, 1)$ such that $\Phi(M) = \Phi(M')$ if M is isomorphic to M' and there are constants $\alpha_1, \beta_1, \alpha_2, \beta_2 \in R$ such that

$$(1) \quad \begin{aligned} \Phi(M) &= 1 && \text{if the ground set of } M \text{ is empty,} \\ \Phi(M) &= \alpha_1 \cdot \Phi(M - e) && \text{if } e \text{ is an isthmus of } M, \\ \Phi(M) &= \beta_1 \cdot \Phi(M - e) && \text{if } e \text{ is a loop of } M, \\ \Phi(M) &= \alpha_2 \cdot \Phi(M/e) + \beta_2 \cdot \Phi(M - e) && \text{otherwise,} \end{aligned}$$

for every matroid M and every element e of M . We also say that Φ is *determined* by the 4-tuple $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. In certain sense (see [7, 2]), all T-G invariants can be reduced from the *Tutte polynomial* of M

$$(2) \quad T(M; x, y) = \sum_{A \subseteq E} (x - 1)^{r(M) - r(A)} (y - 1)^{|A| - r(A)},$$

where E and r denote the ground set and rank function of M , respectively. This is very important invariant that encodes many properties of graphs and has applications in combinatorics, knot theory, statistical physics and coding theory (see cf. [1, 2, 9]).

$M - e = M/e$ if e is a loop or an isthmus of M . Thus the second (third) row of (1) is contained in the fourth row if $\alpha_1 = \alpha_2 + \beta_2$ ($\beta_1 = \alpha_2 + \beta_2$). In this case Φ is called an *isthmus-smooth* (*loop-smooth*) T-G invariant.

We show that any T-G invariant can be transformed to an isthmus- and loop-smooth T-G invariants. The transformations are studied in framework of matroid duality. Furthermore, we discuss modifications of covariance and convolution formulas known for the Tutte polynomial. Notice that transformations into isthmus-smooth invariants are used by decomposition algorithms of T-G invariants in [5].

2. GENERAL MODIFICATIONS

Lemma 1. *Let $\alpha_1, \beta_1, \alpha_2, \beta_2$ be arbitrary elements of a commutative ring $(R, +, \cdot, 0, 1)$. Then $\tilde{T}(M; \alpha_1, \beta_1, \alpha_2, \beta_2) = \alpha_2^{r(M)} \beta_2^{r^*(M)} T(M; \alpha_1/\alpha_2, \beta_1/\beta_2)$ is the unique T - G invariant determined by $(\alpha_1, \beta_1, \alpha_2, \beta_2)$.*

Proof. For any matroid M , denote $\Phi(M) = \tilde{T}(M; \alpha_1, \beta_1, \alpha_2, \beta_2)$ (interpreting the formula as the substitution $x_2 = \alpha_2, y_2 = \beta_2$ in the polynomial $\tilde{T}(M; x_1, y_1, x_2, y_2)$). We use that $(T; x, y)$ is determined by $(x, y, 1, 1)$ and induction on $|E|$. The statement of lemma holds true if $|E| = 0$, otherwise choose $e \in E$. If e is an isthmus of M , then

$$\begin{aligned} \Phi(M) &= \alpha_2^{r(M)} \beta_2^{r^*(M)} T(M; \alpha_1/\alpha_2, \beta_1/\beta_2) = \\ \alpha_2^{r(M-e)+1} \beta_2^{r^*(M-e)} \alpha_1/\alpha_2 T(M-e; \alpha_1/\alpha_2, \beta_1/\beta_2) &= \alpha_1 \Phi(M-e) \end{aligned}$$

by induction hypothesis. If e is a loop of M , then

$$\Phi(M) = \alpha_2^{r(M-e)} \beta_2^{r^*(M-e)+1} \beta_1/\beta_2 T(M-e; \alpha_1/\alpha_2, \beta_1/\beta_2) = \beta_1 \Phi(M-e).$$

If e is neither a loop nor an isthmus of M , then

$$\begin{aligned} \Phi(M) &= \alpha_2^{r(M/e)+1} \beta_2^{r^*(M/e)} T(M/e; \alpha_1/\alpha_2, \beta_1/\beta_2) + \\ \alpha_2^{r(M-e)} \beta_2^{r^*(M-e)+1} T(M-e; \alpha_1/\alpha_2, \beta_1/\beta_2) &= \alpha_2 \Phi(M/e) + \beta_2 \Phi(M-e). \end{aligned}$$

This proves the statement. \square

Lemma 1 also follows from results of Oxley and Welsh [7] (see [2, Corollary 6.2.6]).

Theorem 1. *Let Φ be a T - G invariant determined by $(\alpha_1, \beta_1, \alpha_2, \beta_2)$, $\beta_2 \neq 0$, and $\xi \in R$ be a multiple of β_2 . Then $\Phi_\xi^{\text{is}}(M) = \xi^{|E|} \left(\frac{\alpha_1 - \alpha_2}{\beta_2} \right)^{r^*(M)} \Phi(M)$ is an isthmus-smooth T - G invariant such that for every matroid M ,*

$$\begin{aligned} \Phi_\xi^{\text{is}}(M) &= 1 && \text{if } E = \emptyset, \\ \Phi_\xi^{\text{is}}(M) &= \xi \beta_1 (\alpha_1 - \alpha_2) / \beta_2 \Phi_\xi^{\text{is}}(M-e) && \text{if } e \text{ is a loop of } M, \\ \Phi_\xi^{\text{is}}(M) &= \xi \alpha_2 \Phi_\xi^{\text{is}}(M/e) + \xi (\alpha_1 - \alpha_2) \Phi_\xi^{\text{is}}(M-e) && \text{otherwise.} \end{aligned}$$

Proof. By Lemma 1, $\Phi(M) = \alpha_2^{r(M)} \beta_2^{r^*(M)} T(M; \alpha_1/\alpha_2, \beta_1/\beta_2)$ for each matroid M . Setting $\zeta = \xi \left(\frac{\alpha_1 - \alpha_2}{\beta_2} \right)$ and using equality $|E| = r(M) + r^*(M)$, we get

$$\Phi_\xi^{\text{is}}(M) = \xi^{r(M)} \zeta^{r^*(M)} \Phi(M) = (\xi \alpha_2)^{r(M)} (\zeta \beta_2)^{r^*(M)} T(M; \frac{\xi \alpha_1}{\xi \alpha_2}, \frac{\zeta \beta_1}{\zeta \beta_2}),$$

whence by Lemma 1, Φ_ξ^{is} is a T - G invariant determined by $(\xi \alpha_1, \zeta \beta_1, \xi \alpha_2, \zeta \beta_2)$. Furthermore, $\xi \alpha_1 = \xi \alpha_2 + \zeta \beta_2$, i.e., Φ_ξ^{is} is an isthmus-smooth T - G invariant. \square

Φ_ξ^{is} is called the ξ -isthmus-smooth modification of Φ . Notice that if Φ is an isthmus-smooth invariant (i.e., if $\alpha_1 = \alpha_2 + \beta_2$), then $\Phi_\xi^{\text{is}}(M) = \xi^{|E|} \Phi(M)$ for every matroid M .

If R has zero divisors, then ξ/β_2 does not need to be unique. In this case we should formally replace fraction ξ/β_2 by ξ' where $\xi = \xi' \beta_2$. On the other hand if $\alpha_1 - \alpha_2 = \xi'' \beta_2$,

it suffices to replace fraction $(\alpha_1 - \alpha_2)/\beta_2$ by ξ'' and allow ξ to be any element of R . If R contains no zero divisors, we can extend R into its quotient field and allow ξ to be any element of R , or any element of the quotient field.

If Φ is a T-G invariant determined by $(\alpha_1, \beta_1, \alpha_2, \beta_2)$, then define Φ^* as the T-G invariant determined by $(\beta_1, \alpha_1, \beta_2, \alpha_2)$. Clearly, $\Phi = (\Phi^*)^*$. By Lemma 1, $\Phi(M) = \alpha_2^{r(M)} \beta_2^{r^*(M)} T(M; \alpha_1/\alpha_2, \beta_1/\beta_2)$ and $\Phi^*(M^*) = \beta_2^{r^*(M)} \alpha_2^{r(M)} T(M^*; \beta_1/\beta_2, \alpha_1/\alpha_2)$ for each matroid M . The covariance formula (see [2]) is that $T(M; x, y) = T(M^*; y, x)$, whence

$$(3) \quad \Phi(M) = \Phi^*(M^*).$$

Theorem 2. *Let Φ be a T-G invariant determined by $(\alpha_1, \beta_1, \alpha_2, \beta_2)$, $\alpha_2 \neq 0$, and $\xi \in R$ be a multiple of α_2 . Then $\Phi_\xi^{\text{ls}}(M) = \xi^{|E|} \left(\frac{\beta_1 - \beta_2}{\alpha_2} \right)^{r(M)} \Phi(M)$ is a loop-smooth T-G invariant such that for every matroid M ,*

$$\begin{aligned} \Phi_\xi^{\text{ls}}(M) &= 1 && \text{if } E = \emptyset, \\ \Phi_\xi^{\text{ls}}(M) &= \xi \alpha_1 (\beta_1 - \beta_2) / \alpha_2 \Phi_\xi^{\text{ls}}(M - e) && \text{if } e \text{ is an isthmus of } M, \\ \Phi_\xi^{\text{ls}}(M) &= \xi (\beta_1 - \beta_2) \Phi_\xi^{\text{ls}}(M/e) + \beta_2 \Phi_\xi^{\text{ls}}(M - e) && \text{otherwise.} \end{aligned}$$

Proof. Set $\Phi_\xi^{\text{ls}} = ((\Phi^*)_\xi^{\text{is}})^*$. By (3) and Theorem 1, $\Phi_\xi^{\text{ls}}(M) = ((\Phi^*)_\xi^{\text{is}})^*(M) = (\Phi^*)_\xi^{\text{is}}(M^*)$. Applying Theorem 1 for Φ^* and M^* , we get $(\Phi^*)_\xi^{\text{is}}(M^*) = \xi^{|E|} \left(\frac{\beta_1 - \beta_2}{\alpha_2} \right)^{r(M)} \Phi^*(M^*) = \xi^{|E|} \left(\frac{\beta_1 - \beta_2}{\alpha_2} \right)^{r(M)} \Phi(M)$. Furthermore by definition of Φ^* and Theorem 1, $((\Phi^*)_\xi^{\text{is}})^*$ is determined by $(\xi \alpha_1 (\beta_1 - \beta_2) / \alpha_2, \xi \beta_1, \xi (\beta_1 - \beta_2), \xi \beta_2)$. \square

Notice that $\Phi_\xi^{\text{ls}} = ((\Phi^*)_\xi^{\text{is}})^*$, whence $\Phi_\xi^{\text{is}} = (((\Phi^*)_\xi^{\text{is}})^*)^* = ((\Phi^*)_\xi^{\text{ls}})^*$. Thus

$$(4) \quad \Phi_\xi^{\text{ls}} = ((\Phi^*)_\xi^{\text{is}})^* \text{ and } \Phi_\xi^{\text{is}} = ((\Phi^*)_\xi^{\text{ls}})^*.$$

Φ_ξ^{ls} is called the ξ -loop-smooth modification of Φ . If Φ is an isthmus invariant, then $\Phi_\xi^{\text{ls}}(M) = \xi^{|E|} \Phi(M)$ for every matroid M .

In Theorems 1 and 2 we have assumed that $\beta_2 \neq 0$ and $\alpha_2 \neq 0$, respectively. Let l_M (i_M) denote the number of loops (isthmuses) in a matroid M . If $\alpha_2 = 0$, then by (1), $\Phi(M) = \alpha_1^{r(M)} \beta_1^{l_M} \beta_2^{r^*(M) - l_M}$, whence by (3), $\Phi(M) = \beta_1^{r^*(M)} \alpha_1^{i_M} \alpha_2^{r(M) - i_M}$ if $\beta_2 = 0$. Thus $\Phi(M)$ is easy to evaluate if $\alpha_2 = 0$ or $\beta_2 = 0$ (a contrast with the fact that the Tutte polynomial is difficult to evaluate, see [3, 4, 8]).

3. MODIFICATIONS OF THE TUTTE POLYNOMIAL

Let $\xi \in \mathbb{Z}[x, y]$. Then ξ is a multiple of 1 whence by Theorem 1, the ξ -isthmus-smooth modification of the Tutte polynomial of M is

$$(5) \quad T_\xi^{\text{is}}(M; x, y) = \xi^{|E|} (x - 1)^{r^*(M)} T(M; x, y)$$

and satisfies

$$(6) \quad \begin{aligned} T_{\xi}^{\text{is}}(M; x, y) &= 1 && \text{if } E = \emptyset, \\ T_{\xi}^{\text{is}}(M; x, y) &= \xi y(x-1)T_{\xi}^{\text{is}}(M-e; x, y) && \text{if } e \text{ is a loop of } M, \\ T_{\xi}^{\text{is}}(M; x, y) &= \xi T_{\xi}^{\text{is}}(M/e; x, y) + \xi(x-1)T_{\xi}^{\text{is}}(M-e; x, y) && \text{otherwise.} \end{aligned}$$

By Theorem 2, the ξ -loop-smooth modification of the Tutte polynomial of M is

$$(7) \quad T_{\xi}^{\text{ls}}(M; x, y) = \xi^{|E|}(y-1)^{r(M)}T(M; x, y)$$

and satisfies

$$(8) \quad \begin{aligned} T_{\xi}^{\text{ls}}(M; x, y) &= 1 && \text{if } E = \emptyset, \\ T_{\xi}^{\text{ls}}(M; x, y) &= \xi x(y-1)T_{\xi}^{\text{ls}}(M-e; x, y) && \text{if } e \text{ is an isthmus,} \\ T_{\xi}^{\text{ls}}(M; x, y) &= \xi(y-1)T_{\xi}^{\text{ls}}(M-e; x, y) + \xi T_{\xi}^{\text{ls}}(M/e; x, y) && \text{otherwise.} \end{aligned}$$

By (3), $T_{\xi}^{\text{ls}}(M; x, y) = (T_{\xi}^{\text{ls}})^*(M^*; x, y)$, and by (4), $(T_{\xi}^{\text{ls}})^*(M^*; x, y) = (T^*)_{\xi}^{\text{is}}(M^*; x, y)$. By (2) and (1), we have $T^*(M^*; x, y) = T(M^*; y, x)$, whence $(T^*)_{\xi}^{\text{is}}(M^*; x, y) = T_{\xi}^{\text{is}}(M^*; y, x)$, i.e., we have a variant of the covariance formula

$$(9) \quad T_{\xi}^{\text{ls}}(M; x, y) = T_{\xi}^{\text{is}}(M^*; y, x).$$

Kook, Reiner, and Stanton [6] introduced the convolution formula

$$T(M; x, y) = \sum_{A \subseteq E} T(M/A; x, 0) \cdot T(M|A; 0, y),$$

(where $M|A$ and M/A denote the restriction of M to A and the contraction of A from M , respectively). Hence by (5) and (7),

$$T(M; x, y) = \sum_{A \subseteq E} \xi^{-|E|}(-1)^{-r(M/A)}T_{\xi}^{\text{ls}}(M/A; x, 0) \cdot (-1)^{-r^*(M|A)}T_{\xi}^{\text{is}}(M|A; 0, y).$$

Since $r^*(M|A) = |A| - r(A)$, $r(M/A) = r(M) - r(A)$, and $2r(A) - r(M) - |A|$ has the same parity as $r(M) + |A|$, we get a variant of the convolution formula

$$(10) \quad T(M; x, y) = \xi^{-|E|}(-1)^{r(M)} \sum_{A \subseteq E} (-1)^{|A|}T_{\xi}^{\text{ls}}(M/A; x, 0) \cdot T_{\xi}^{\text{is}}(M|A; 0, y).$$

The ring $\mathbb{Z}[x, y]$ has no divisors of zero, therefore it has a quotient field $\mathbb{F}[x, y]$, consisting of all rational polynomials with integral coefficients. Thus, as pointed out in the remark after Theorem 1, for any $\xi \in \mathbb{F}[x, y]$, we can consider ξ -isthmus- and ξ -loop-smooth modifications of the Tutte polynomial, thus also formulas (9), (10).

If Φ is a T-G invariant determined by $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ and $\xi, \zeta \in R$, then by Lemma 1, there exists a T-G invariant determined by $(\xi\alpha_1, \zeta\beta_1, \xi\alpha_2, \zeta\beta_2)$ denoted by $\Phi_{\xi, \zeta}$. Clearly, $\Phi_{\xi, \zeta}(M) = \xi^{r(M)}\zeta^{r^*(M)}\Phi(M)$ for each matroid M . Suppose that $\Phi_{\xi, \zeta}$ is isthmus- and loop-smooth in the same time. Then $\xi\alpha_1 = \zeta\beta_1 = \xi\alpha_2 + \zeta\beta_2$, whence $\xi/\zeta = \beta_1/\alpha_1 = \beta_2/(\alpha_1 - \alpha_2) = (\beta_1 - \beta_2)/\alpha_2$, and thus

$$(11) \quad \beta_1 = \alpha_1\beta_2/(\alpha_1 - \alpha_2).$$

On the other hand (11) implies $\beta_1/\alpha_1 = \beta_2/(\alpha_1 - \alpha_2)$ and $(\beta_1 - \beta_2)/\alpha_2 = \beta_2/(\alpha_1 - \alpha_2)$. Thus (11) is a necessary and sufficient condition for existence of ξ and ζ such that $\Phi_{\xi,\zeta}$ is an isthmus- and loop-smooth invariant. Therefore this kind of transformation cannot be applied for each Φ . In particular, (11) is not valid for the Tutte polynomial because $y \neq x/(x - 1)$.

REFERENCES

- [1] L. Beaudin, J. Ellis-Monaghan, G. Pangborn, and R. Shrock, *A little statistical mechanics for the graph theorist*, Discrete Math. **310** (2010) 2037–2053.
- [2] T. Brylawski and J. Oxley, *The Tutte polynomial and its applications*, in: Matroid Applications, (N. White, Editor), Cambridge University Press, Cambridge (1992), pp. 123–225.
- [3] L.A. Goldberg and M. Jerrum, *Inapproximability of the Tutte polynomial*, Inform. and Comput. **206** (2008) 908–929.
- [4] F. Jaeger, D.L. Vertigan, and D.J.A. Welsh, *On the computational complexity of the Jones and Tutte polynomials*, Math. Proc. Cambridge Philos. Soc. **108** (1990) 35–53.
- [5] M. Kochol, *Splitting formulas for Tutte-Grothendieck invariants*, manuscript (2014).
- [6] W. Kook, V. Reiner, and D. Stanton, *A convolution formula for the Tutte polynomial*, J. Combin. Theory Ser. B **76** (1999) 297–300.
- [7] J.G. Oxley and D.J.A. Welsh, *The Tutte polynomial and percolation*, in: Graph Theory and Related Topics, (J.A. Bondy and U.S.R. Murty, Editors), Academic Press, New York (1979), pp. 329–339.
- [8] D. Vertigan, *The computational complexity of Tutte invariants for planar graphs*, SIAM J. Comput. **135** (2005) 690–712.
- [9] D. J. A. Welsh, *Complexity: Knots, Colourings and Counting*, London Math. Soc. Lecture Notes Series 186, Cambridge University Press, Cambridge (1993).

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